A Practical Model for the Calculation of Multiply Scattered Lidar Returns–Errata and Extensions to Applied Optics paper: 1998, **37**, 2464-2474.

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1 Errata-equation 16

From the AO paper, page 2469:

$$\frac{P_n(R)}{P_1(R)} < \frac{\mathcal{P}_{n\pi}}{\mathcal{P}(\pi)} \frac{(\beta d)^{n-1}}{(n-1)!} \left\{ 1 - exp\left(-\frac{\rho_t^2 R^2}{\Theta_s^2 d^2}\right) + \frac{\sqrt{\pi}\rho_t R}{\Theta_s d} \left[1 - erf\left(\frac{\rho_t R}{\Theta_s d}\right)\right]\right\} \quad (15)$$

This can be summed over all orders of scattering when $\frac{\mathcal{P}_{n\pi}(R)}{\mathcal{P}(\pi,R)} = 1$:

$$\frac{P_t(R)}{P_1(R)} < e^{\tau} \left\{ 1 - exp\left(-\frac{\rho_t^2 R^2}{\Theta_s^2 d^2}\right) + \frac{\sqrt{\pi}\rho_t R}{\Theta_s d} \left[1 - erf\left(\frac{\rho_t R}{\Theta_s d}\right)\right]\right\} \quad (16)$$

The summation of terms from equation 15 to include all orders of scattering is incorrect. The single scattering contribution should not be multiplied by the factor contained in curly brackets. The correct statement of equation 16 is:

$$\frac{P_t(R)}{P_1(R)} < 1 + (e^{\tau} - 1) \left\{ 1 - exp\left(-\frac{\rho_t^2 R^2}{\Theta_s^2 d^2}\right) + \frac{\sqrt{\pi}\rho_t R}{\Theta_s d} \left[1 - erf\left(\frac{\rho_t R}{\Theta_s d}\right)\right]\right\} \quad (16)$$

Thanks to Dave Whiteman of NASA Goddard for pointing out that the published version of equation 16 generates unphysical results.

2 Simplifications to equation 11

Equation 11 of the AO paper provides the ratio of nth order scattering to single scattering:

$$\frac{P_n(R)}{P_1(R)} = \frac{\mathcal{P}_{n\pi}(R)}{\mathcal{P}(\pi,R)}\left[1 - exp(-\frac{\rho_t^2}{\rho_l^2})\right]^{-1}\left[\frac{\tau^{n-1}}{(n-1)!} - \frac{1}{2^{n-1}}\int_{-d}^{d}\beta_s(x_1)\int_{x_1}^{d}\beta_s(x_2)\int_{x_2}^{d}\beta_s(x_3)\cdots \int_{x_{n-3}}^{d}\beta_s(x_{n-2})\int_{x_{n-2}}^{d}\beta_s(x_{n-1}) exp\left(-\frac{\rho_t^2 R^2}{x_1^2\Theta_s^2(x_1) + x_2^2\Theta_s^2(x_2) + \cdots + x_{n-1}^2\Theta_s^2(x_{n-1}) + \rho_l^2 R^2}\right) dx_1 dx_2 dx_3 \cdots dx_{n-1}\right] \quad (11)$$

This equation is correct. However, it can be simplified by expressing it as a sum of terms resulting from splitting the limits of integration into two parts: $-d \to 0$ and $0 \to d$. Notice that when the outside integral is split, the limits of the inner integrals extend outside the new limits. Splitting the inner limits into the same domains produces additional terms. Using the fact that the kernel of the integral is symmetric about zero makes it possible to identify all of the terms within the limits $0 \to d$ as identical to terms from within the limits $-d \to 0$. When these terms are summed the following expression results:

$$\frac{P_n(R)}{P_1(R)} = \frac{\mathcal{P}_{n\pi}(R)}{\mathcal{P}(\pi,R)}\left[1 - exp(-\frac{\rho_t^2}{\rho_l^2})\right]^{-1}\left[\frac{\tau^{n-1}}{(n-1)!} - \int_{-d}^{0}\beta_s(x_1)\int_{x_1}^{0}\beta_s(x_2)\int_{x_2}^{0}\beta_s(x_3)\cdots \int_{x_{n-3}}^{0}\beta_s(x_{n-2})\int_{x_{n-2}}^{d}\beta_s(x_{n-1}) exp\left(-\frac{\rho_t^2 R^2}{x_1^2\Theta_s^2(x_1) + x_2^2\Theta_s^2(x_2) + \cdots + x_{n-1}^2\Theta_s^2(x_{n-1}) + \rho_l^2 R^2}\right) dx_1 dx_2 dx_3 \cdots dx_{n-1}\right] \quad (12)$$

While these new limits reduce the computational effort required to perform the integration, having the origin located at the point of the backscatter event is inconvenient.

Rewriting equation 12 in terms of distance from the lidar, $r = x + R$ with the origin at the lidar and R equal to the distance to the single scatter event yields:

$$\frac{P_n(R)}{P_1(R)} = \frac{\mathcal{P}_{n\pi}(R)}{\mathcal{P}(\pi, R)} \left[1 - exp(-\frac{\rho_t^2}{\rho_l^2})\right]^{-1} \left[\frac{\tau^{n-1}}{(n-1)!} - \int_{r_c}^{R} \beta_s(r_1) \int_{r_1}^{R} \beta_s(r_2) \int_{r_2}^{R} \beta_s(r_3) \cdots \int_{r_{n-3}}^{R} \beta_s(r_{n-2}) \int_{r_{n-2}}^{R} \beta_s(r_{n-1}) exp\left(-\frac{\rho_t^2 R^2}{(R-r_1)^2\Theta_s^2(r_1) + (R-r_2)^2\Theta_s^2(r_2) + \cdots + (R-r_{n-1})^2\Theta_s^2(r_{n-1}) + \rho_l^2 R^2}\right) dr_1 dr_2 dr_3 \cdots dr_{n-1}\right] \quad (13)$$

Where r_c is the distance to the cloud base.